Full characterization of the fractional Poisson process

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The fractional Poisson process (FPP) is a counting process with independent and identically distributed inter-event times following the Mittag-Leffler distribution. This process is very useful in several fields of applied and theoretical physics including models for anomalous diffusion. Contrary to the well-known Poisson process, the fractional Poisson process does not have stationary and independent increments. It is not a Lévy process and it is not a Markov process. In this letter, we present formulae for its finite-dimensional distribution functions, fully characterizing the process. These exact analytical results are compared to Monte Carlo simulations.

From a loose mathematical point of view, counting processes $N(A)$ are stochastic processes that count the random number of points in a set A. They are used in many fields of physics and other applied sciences. In this letter, we will consider one-dimensional real sets with the physical meaning of time intervals. The points will be incoming events whose duration is much smaller than the inter-event or inter-arrival waiting time. For instance, counts from a Geiger-Müller counter can be described in this way. The number of counts, $N(\Delta t)$, in a given time interval Δt is known to follow the Poisson distribution

$$\mathbb{P}(N(\Delta t) = n) = \exp(-\lambda \Delta t)\frac{(\lambda \Delta t)^n}{n!}, \qquad (1)$$

where λ is the constant rate of arrival of ionizing particles. Together with the assumption of independent and stationary increments, Eq. (1) is sufficient to define the *homogeneous* Poisson process. Curiously, one of the first occurrences of this process in the scientific literature was connected to the number of casualties by horse kicks in the Prussian army cavalry [1]. The Poisson process is strictly related to the exponential distribution. The inter-arrival times τ_i identically follow the exponential distribution and are independent random variables. This means that the Poisson process is a prototypical *renewal* process. A justification for the ubiquity of the Poisson process has to do with its relationship with the binomial distribution. Suppose that the time interval of interest $(t, t + \Delta t)$ is divided into n equally spaced sub-intervals. Further assume that a counting event appears in such a sub-interval with probability p and does not appear with probability $1 - p$. Then, $\mathbb{P}(N(\Delta t) = k) = \mathrm{Bin}(k; p, n)$ is a binomial distribution of parameters p and n and the expected number of events in the time interval is given by $\mathbb{E}[N(\Delta t)] = np$. If this expected number is kept constant for $n \to \infty$, the binomial distribution converges to the Poisson distribution of parameter $\lambda = \mathbb{E}[N(\Delta t)]/\Delta t$, while, in the meantime, $p \to 0$. However, it can be shown that many counting processes with non-stationary increments converge to the Poisson process after a transient period. It is sufficient to require that they are renewal process (i.e. they have independent and identically distributed (iid) inter-arrival times) and that $\mathbb{E}(\tau_i) < \infty$. In other words, many counting processes with non-independent and non-stationary increments behave as the Poisson process if observed long after the transient period.

In recent times, it has been shown that heavy-tailed distributed inter-arrival times (for which $\mathbb{E}(\tau_i) = \infty$) do play a role in many phenomena such as blinking nano-dots [2, 3], human dynamics [4, 5] and the related inter-trade times in financial markets [6, 7].

Among the counting processes with non-stationary increments, the so-called *fractional Poisson process* [8], $N_\beta(t)$, is particularly important because it is the thinning limit of counting processes related to renewal processes with power-law distributed inter-arrival times [9, 10]. Moreover, it can be used to approximate anomalous diffusion ruled by space-time fractional diffusion equations [9, 11–16]. It is a straightforward generalization of the Poisson process defined as follows. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed positive random variables with the meaning of inter-arrival times and let their common cumulative distribution function (cdf) be

$$F_\tau(t) = \mathbb{P}(\tau \le t) = 1 - E_\beta(-t^\beta), \qquad (2)$$

where $E_\beta(-t^\beta)$ is the one-parameter Mittag-Leffler function, $E_\beta(z)$, defined in the complex plane as

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)} \qquad (3)$$

evaluated in the point $z = -t^\beta$ and with the prescription $0 < \beta \le 1$. In equation (3), $\Gamma(\cdot)$ is Euler's Gamma

function. The sequence of the *epochs*, $\{T_n\}_{n=1}^{\infty}$, is given by the sums of the inter-arrival times

$$T_n = \sum_{i=1}^{n} \tau_i. \tag{4}$$

The epochs represent the times in which events arrive or occur. Let $f_\tau(t) = dF_\tau(t)/dt$ denote the probability density function (pdf) of the inter-arrival times, then the probability density function of the n-th epoch is simply given by the n-fold convolution of $f_\tau(t)$, written as $f_\tau^{*n}(t)$. In Ref. [10], it is shown that

$$f_{T_n}(t) = f_\tau^{*n}(t) = \beta \frac{t^{n\beta-1}}{(n-1)!} E_\beta^{(n)}(-t^\beta), \tag{5}$$

where $E_\beta^{(n)}(-t^\beta)$ is the n-th derivative of $E_\beta(z)$ evaluated in $z = -t^\beta$. The counting process $N_\beta(t)$ counts the number of epochs (events) up to time t, assuming that $T_0 = 0$ is an epoch as well, or, in other words, that the process begins from a renewal point. This assumption will be used all over this paper. $N_\beta(t)$ is given by

$$N_\beta(t) = \max\{n : \; T_n \leq t\}. \tag{6}$$

In Ref. [9], the fractional Poisson distribution is derived and it is given by

$$\mathbb{P}(N_\beta(t) = n) = \frac{t^{\beta n}}{n!} E_\beta^{(n)}(-t^\beta). \tag{7}$$

Eq. (7) coincides with the Poisson distribution of parameter $\lambda = 1$ for $\beta = 1$. In principle, equations (3) and (7) can be directly used to derive the fractional Poisson distribution, but convergence of the series is slow. Fortunately, in a recent paper, Beghin and Orsingher proved that

$$E_\beta^{(n)}(-t^\beta) = \\ \frac{n!}{t^{\beta n}} \int_0^\infty F_{S_\beta}(t; u) \left[\frac{\exp(-u)u^{n-1}}{(n-1)!} - \frac{\exp(-u)u^n}{n!} \right] du, \tag{8}$$

where $F_{S_\beta}(t; u)$ is the cdf of a stable random variable $S_\beta(\nu, \gamma, \delta)$ with index β, skewness parameter $\nu = 1$, scale parameter $\gamma = (u \cos \pi\beta/2)^{1/\beta}$ and location $\delta = 0$ [17]. The integral in equation (8) can be evaluated numerically and Fig. 11 shows $\mathbb{P}(N_\beta(t) = n)$ for three different values of β. The Monte Carlo simulation of the fractional Poisson process is based on the algorithm presented in equation (20) of Ref. [14].

As a consequence of Kolmogorov's extension theorem, in order to fully characterize the stochastic process $N_\beta(t)$, one has to derive its finite dimensional distributions. A further requirement on the process' paths uniquely determines the process, namely that they are right-continuous step functions with left limits [18]. The finite-dimensional

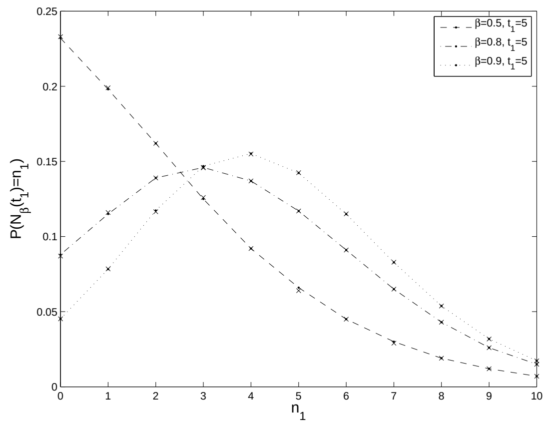

Figure 1: $P(N_\beta(T_1) = n_1)$ as function of n_1 for three different values of β. The crosses are estimations obtained from 10^5 Monte Carlo samples and the lines are given to guide the eye.

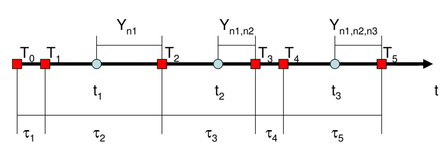

Figure 2: (Color online) Pictorial illustration of the random variables used in the text. The light blue dots represent the observation points t_1, t_2 and t_3. The red squares are the epochs $T_0 = 0, T_1, \ldots, T_5$. The conditional residual life-time is the time elapsed between t_i and the next epoch T_{n_i+1}. It depends on previous values of n_i, this is the number of events between 0 and t_i, with the event at $t = T_0 = 0$ not considered. Here, we have $n_1 = 1$, $n_2 = 2$ and $n_3 = 4$. All the equations in this paper can be derived by analyzing this figure.

distributions are the multivariate probability distribution functions $\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2, \ldots, N_\beta(t_k) = n_k)$ with $t_1 < t_2 < \ldots < t_k$ and $n_1 \leq n_2 \leq \ldots \leq n_k$. We have already given the formula for the one-point functions in Eq. (7). The general finite dimensional distribution can be computed observing that the event $\{N_\beta(t_1) = n_1, N_\beta(t_2) = n_2, \ldots, N_\beta(t_k) = n_k\}$ is equivalent to $\{0 < T_{n_1} < t_1, T_{n_1+1} > t_1, t_1 < T_{n_2} < t_2, T_{n_2+1} > t_2, \ldots, t_{k-1} < T_{n_k} < t_k, T_{n_k+1} > t_k\}$. Therefore, we find

$$\begin{aligned} &\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2, \ldots, N_\beta(t_k) = n_k) = \\ &\mathbb{P}(0 < T_{n_1} < t_1, T_{n_1+1} > t_1, t_1 < T_{n_2} < t_2, T_{n_2+1} > t_2, \\ &\ldots, t_{k-1} < T_{n_k} < t_k, T_{n_k+1} > t_k) = \\ &\int_0^{t_1} du_1 f_\tau^{*n_1}(u_1) \int_{t_1-u_1}^{\infty} du_2 f_\tau(u_2) \\ &\int_{t_1-u_1-u_2}^{t_2-u_1-u_2} du_3 f_\tau^{*(n_2-n_1-1)}(u_3) \int_{t_2-u_1-u_2-u_3}^{\infty} du_4 f_\tau(u_4) \\ &\ldots \int_{t_{k-1}-\sum_{i=1}^{2k-2} u_i}^{t_k-\sum_{i=1}^{2k-2} u_i} du_{2k-1} f_\tau^{*(n_k-n_{k-1}-1)}(u_{2k-1}) \\ &\left[1 - F_\tau\left(t_k - \sum_{i=1}^{2k-1} u_i\right)\right]. \end{aligned} \tag{9}$$

For instance, the two point function is given by

$$\begin{aligned} &\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2) = \\ &\mathbb{P}(0 < T_{n_1} < t_1, T_{n_1+1} > t_1, t_1 < T_{n_2} < t_2, T_{n_2+1} > t_2) = \\ &\int_0^{t_1} du_1 f_\tau^{*n_1}(u_1) \int_{t_1-u_1}^{\infty} du_2 f_\tau(u_2) \\ &\int_{t_1-u_1-u_2}^{t_2-u_1-u_2} du_3 f_\tau^{*(n_2-n_1-1)}(u_3) \\ &\left[1 - F_\tau\left(t_2 - u_1 - u_2 - u_3\right)\right]. \end{aligned} \tag{10}$$

Let us focus on the two-point case for the sake of illustration. As $N_\beta(t)$ is a counting process, one has $\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2) = \mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1)$ and, as a consequence of the definition of conditional probability

$$\begin{aligned} &\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1) = \\ &\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1) \times \\ &\times \mathbb{P}(N_\beta(t_1) = n_1). \end{aligned} \tag{11}$$

For $\beta = 1$, when the fractional Poisson process coincides with the standard Poisson process, the increments are iid random variables and one has

$$\begin{aligned} &\mathbb{P}(N_1(t_2) - N_1(t_1) = n_2 - n_1 | N_1(t_1) = n_1) = \\ &\mathbb{P}(N_1(t_2) - N_1(t_1) = n_2 - n_1) = \\ &\exp(-(t_2 - t_1)) \frac{(t_2 - t_1)^{(n_2-n_1)}}{(n_2 - n_1)!}. \end{aligned} \tag{12}$$

On the contrary, for $0 < \beta < 1$, the increment $N_\beta(t_2) - N_\beta(t_1)$ and $N_\beta(t_1)$ are not independent. Note that $N_\beta(t_1)$ can be seen as an increment as $N_\beta(0) = 0$ by definition. However from Eq. (11), the conditional probability of having $n_2 - n_1$ epochs in the interval (t_1, t_2) conditional on the observation of n_1 epochs in the interval $(0, t_1)$ can be written as a ratio of two finite dimensional distribution:

$$\begin{aligned} &\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1) = \\ &\frac{\mathbb{P}(N_\beta(t_1) = n_1, N_\beta(t_2) = n_2)}{\mathbb{P}(N_\beta(t_1) = n_1)}. \end{aligned} \tag{13}$$

This probability can be evaluated by means of an alternative method, more appealing for a direct and practical understanding of the dependence structure. Let

$$Y_{n_1} \stackrel{\text{def}}{=} [T_{n_1+1} - t_1 | N_\beta(t_1) = n_1] \tag{14}$$

denote the residual lifetime at time t_1 (that is the time to the next epoch or renewal) conditional on $N_\beta(t_1) = n_1$. With reference to Fig. 2, one can see that the conditional probability $\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1)$ is given by the following convolution integral for $n_2 - n_1 \geq 1$

$$\begin{aligned} &\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = n_2 - n_1 | N_\beta(t_1) = n_1) = \\ &\int_0^{t_2-t_1} \mathbb{P}(N_\beta(t_2 - t_1 - y) = n_2 - n_1 - 1) f_{Y_{n_1}}(y)\, dy, \end{aligned} \tag{15}$$

where $f_{Y_{n_1}}(t)$ is the pdf of Y_{n_1}. In the case $n_2 - n_1 = 0$, one has

$$\mathbb{P}(N_\beta(t_2) - N_\beta(t_1) = 0 | N_\beta(t_1) = n_1) = 1 - F_{Y_{n_1}}(t_2 - t_1) \tag{16}$$

where $F_{Y_{n_1}}(y)$ is the cdf of Y_{n_1}. The distribution of the conditional residual lifetime Y_{n_1} can be evaluated in several ways. For instance, one can notice that it can be decomposed as follows

$$Y_{n_1} = \tilde{\tau}_{n_1+1} + U_{n_1} \tag{17}$$

where U_{n_1} is defined as

$$U_{n_1} \stackrel{\text{def}}{=} [T_{n_1} | N_\beta(t_1) = n_1], \tag{18}$$

and is the position of the last epoch before t_1 conditional on $N_\beta(t_1) = n_1$, and

$$\tilde{\tau}_{n_1+1} \stackrel{\text{def}}{=} [\tau_{n_1+1} - t_1 | T_{n_1+1} > t_1] \tag{19}$$

is the difference between τ_{n_1+1} and t_1 conditional on $T_{n_1+1} > t_1$. The pdf of U_{n_1} is given by the following

chain of equalities

$$\begin{aligned}f_{U_{n_1}}(t)dt =&\mathbb{P}(T_{n_1}\in dt|N_\beta(t_1)=n_1)\\ =&\mathbb{P}(T_{n_1}\in dt|T_{n_1}<t_1,T_{n_1}+\tau_{n_1+1}>t_1)\\ =&\mathbb{P}(T_{n_1}\in dt|T_{n_1}<t_1,\tau_{n_1+1}>t_1-T_{n_1})\\ \overset{\star}{=}&\frac{\mathbb{P}(T_{n_1}\in dt)\int_{t_1-t}^{\infty}\mathbb{P}(\tau_{n_1+1}\in dw)}{\mathbb{P}(T_{n_1}<t_1,\tau_{n_1+1}>t_1-T_{n_1})}\\ \overset{*}{=}&\frac{f_\tau^{*n_1}(t)[1-F_\tau(t_1-t)]dt}{\int_0^{t_1}du f_\tau^{*n_1}(u)[1-F_\tau(t_1-u)]},\end{aligned} \tag{20}$$

where we used the independence between T_{n_1} and τ_{n_1+1} ($\star$) and $f_{T_{n_1}}(x)=f_\tau^{*n_1}(x)$ ($*$). The pdf of $\tilde{\tau}_{n_1+1}$ is

$$\begin{aligned}f_{\tilde{\tau}_{n_1+1}}(t|U_{n_1})dt =&\mathbb{P}(\tau_{n_1+1}-t_1\in dt|T_{n_1+1}>t_1)\\ =&\frac{\mathbb{P}(\tau_{n_1+1}\in dt+t_1)}{\mathbb{P}(\tau_{n_1+1}>t_1-U_{n_1})}\\ =&\frac{f_\tau(t+t_1)dt}{1-F_\tau(t_1-U_{n_1})}.\end{aligned} \tag{21}$$

From Eq. (17), one can write that

$$f_{Y_{n_1}}(t)=\int_0^{t_1}f_{\tilde{\tau}_{n_1+1}}(t-u|u)f_{U_{n_1}}(u)du \tag{22}$$

and this equation leads to

$$f_{Y_{n_1}}(t)=\frac{\int_0^{t_1}du f_\tau^{*n_1}(u)f_\tau(t+t_1-u)}{\int_0^{t_1}du f_\tau^{*n_1}(u)[1-F_\tau(t_1-u)]} \tag{23}$$

that, together with Eq. (7), gives us the probability of the conditional increments in Eq. (15). Notice that, for $n_1=0$, one has $f_\tau^{*0}(u)=\delta(u)$ and Eq. (23) reduces to the familiar equation for the residual life-time pdf in the absence of previous renewals

$$f_{Y_0}(t)=\frac{f_\tau(t+t_1)}{1-F_\tau(t_1)}. \tag{24}$$

This method can be applied to the general multidimensional case. As in Eq. (11) we can write

$$\begin{aligned}&\mathbb{P}(N_\beta(t_1)=n_1,\ldots,N_\beta(t_k)=n_k,N_\beta(t_{k+1})=n_{k+1})=\\ &\quad\mathbb{P}(N_\beta(t_{k+1})-N_\beta(t_k)=n_{k+1}-n_k|\\ &\quad N_\beta(t_1)=n_1,\ldots,N_\beta(t_k)=n_k)\times\\ &\quad\times\mathbb{P}(N_\beta(t_1)=n_1,\ldots,N_\beta(t_k)=n_k)\end{aligned} \tag{25}$$

and the predictive probabilities can be evaluated as

$$\begin{aligned}&\mathbb{P}(N_\beta(t_{k+1})-N_\beta(t_k)=n_{k+1}-n_k|\ldots\\ &\quad|N_\beta(t_1)=n_1,\ldots,N_\beta(t_k)=n_k)=\\ &\int_0^{t_{k+1}-t_k}\mathbb{P}(N_\beta(t_{k+1}-t_k-y)=n_{k+1}-n_k-1)\times\\ &\quad\times f_{Y_{n_1,\ldots,n_k}}(y)dy,\end{aligned} \tag{26}$$

where we defined

$$Y_{n_1,\ldots,n_k}\overset{\text{def}}{=}[T_{n_k+1}-t_k|N_\beta(t_1)=n_1,\ldots,N_\beta(t_k)=n_k]. \tag{27}$$

Again, we can use a decomposition of $Y_{n_1,\ldots,n_k}$

$$Y_{n_1,\ldots,n_k}=\tilde{\tau}_{n_k+1}+U_{n_k}, \tag{28}$$

where

$$U_{n_k}\overset{\text{def}}{=}[T_{n_k}|N_\beta(t_1)=n_1,\ldots,N_\beta(t_k)=n_k], \tag{29}$$

and

$$\tilde{\tau}_{n_k+1}\overset{\text{def}}{=}[\tau_{n_k+1}-t_k|T_{n_k+1}>t_k]. \tag{30}$$

The difference with the two-point case is that $U_{n_1}=[T_{n_1}|N_\beta(t_1)=n_1]=[\sum_{i=1}^{n_1}\tau_i|N_\beta(t_1)=n_1]$ must be replaced by

$$U_{n_k}=t_{k-1}+Y_{n_1,\ldots,n_{k-1}}+\left[\sum_{i=n_{k-1}+1}^{n_k}\tau_i|N_\beta(t_k)=n_k\right]. \tag{31}$$

The time between t_{k-1} and the next renewal epoch is $Y_{n_1,\ldots,n_{k-1}}$ and it is independent from $\sum_{i=n_{k-1}+1}^{n_k}\tau_i$. Therefore, the convolution

$$q(n_1,\ldots,n_k;t)=f_{Y_{n_1,\ldots,n_{k-1}}}*f_\tau^{*(n_k-n_{k-1}-1)}(t) \tag{32}$$

replaces $f_\tau^{*n_1}(t)$ in Eq. (20). This leads to

$$\begin{aligned}&f_{U_{n_k}}(z)=\\ &\frac{q(n_1,\ldots,n_k;t+t_{k-1})[1-F_\tau(t_k-t)]}{\int_{t_{k-1}}^{t_k}q(n_1,\ldots,n_k;u+t_{k-1})[1-F_\tau(t_k-u)]du}.\end{aligned} \tag{33}$$

On the other hand, $f_{\tilde{\tau}_{n_k+1}}(t)$ has the same functional form as $f_{\tilde{\tau}_{n_1+1}}(t)$ given in Eq. (21) with U_{n_k} replacing U_{n_1}. Therefore, $Y_{n_1,\ldots,n_k}$ has the following pdf

$$\begin{aligned}&f_{Y_{n_1,\ldots,n_k}}(t)=\\ &\frac{\int_{t_{k-1}}^{t_k}du\,q(n_1,\ldots,n_k;u+t_{k-1})f_\tau(t+t_k-u)}{\int_{t_{k-1}}^{t_k}du\,q(n_1,\ldots,n_k;u+t_{k-1})[1-F_\tau(t+t_k-u)]}.\end{aligned} \tag{34}$$

In practice, the random variable $Y_{n_1,\ldots,n_{k-1}}$ carries the memory of the observations made at times $t_1,\ldots,t_{k-1}$; the knowledge of $f_{Y_{n_1,\ldots,n_{k-1}}}$ allows the computation of $f_{Y_{n_1,\ldots,n_k}}$, and, via Eqs. (25) and (26), the $k+1$-dimensional distribution can be derived as well.

Figs. 3 and 4 compare the theoretical results of Eqs. (20), (23) and (24) with those of a Monte Carlo simulation based on the algorithm presented in equation (20) of Ref. [14].

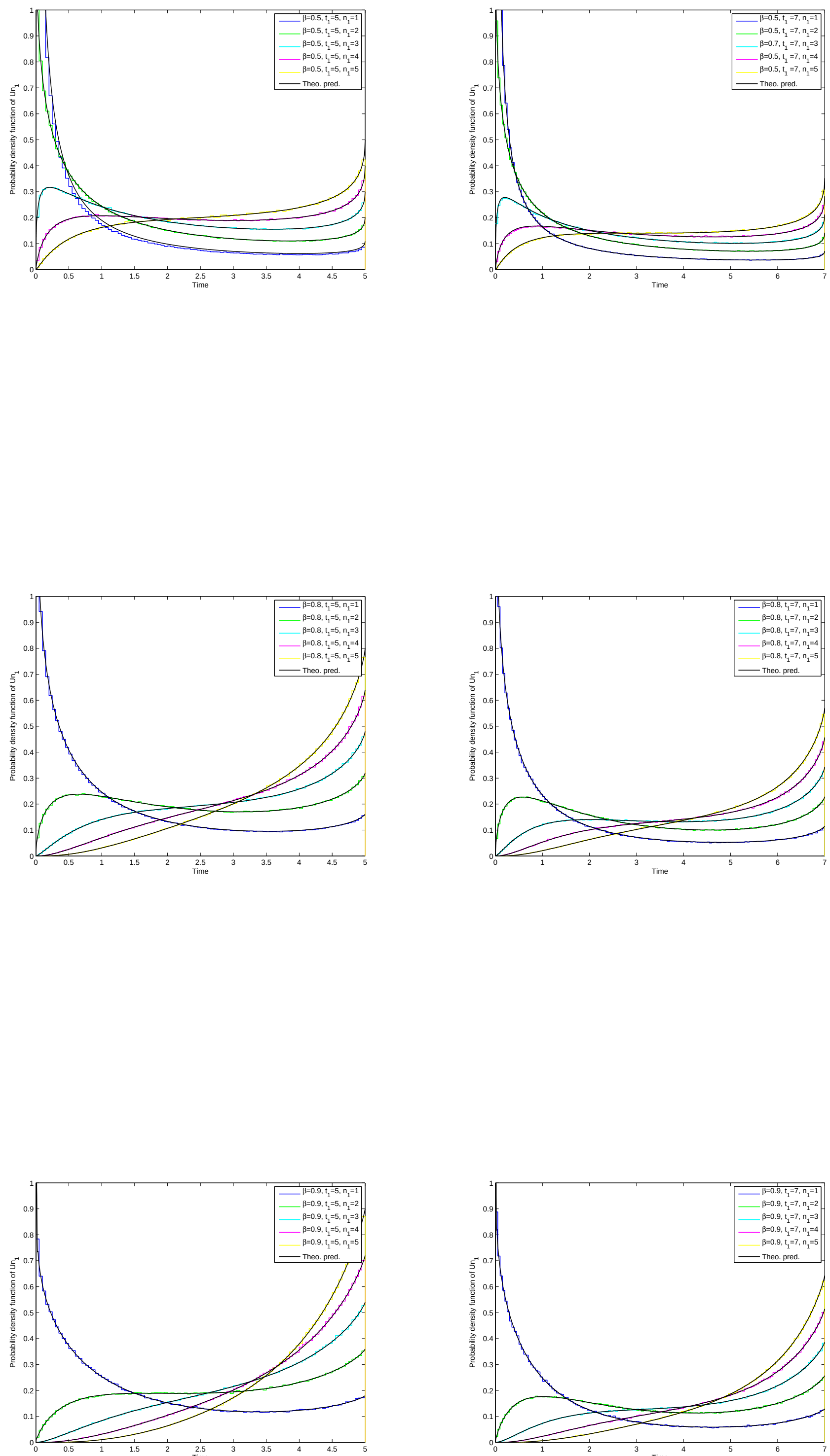

Figure 3: (Color online) Pdf of the random variable U_{n_1} as given in Eq. (20) (solid black lines) compared to Monte Carlo simulations (colored step lines) for three values of β and two different values of t_1. 10^7 different paths were simulated for each value of β and the bin width is 0.05. Time is in arbitrary units.

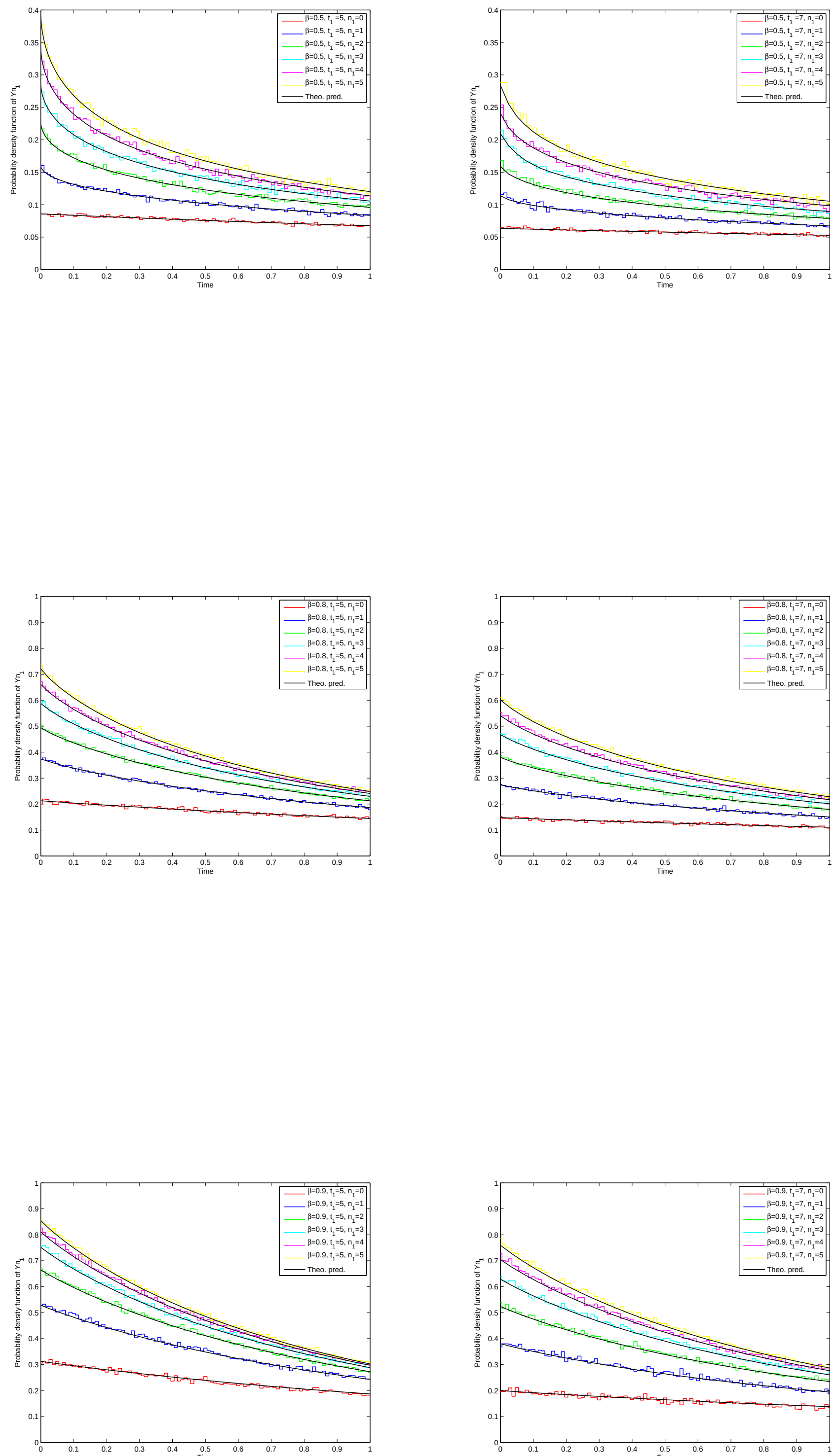

Figure 4: (Color online) Pdf of the random variable Y_{n_1} as given in Eqs. (23) and (24) (solid black lines) compared to Monte Carlo simulations (colored step lines) for three values of β and two different values of t_1. 10^7 different paths were simulated for each value of β and the bin width is 0.01. Time is in arbitrary units.

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